Inequality

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Let a and b be positive real numbers satisfying

$a+b \geq (a-b)^2$. Prove that $x^a(1-x)^b + x^b(1-x)^a \leq \dfrac{1}{2^{a+b-1}}$ for any real $x \in (0,1)$.

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Since $x^a(1-x)^b + x^b(1-x)^a \leq (1/2)^{a+b-1} \Leftrightarrow (2x)^a(2-2x)^b + (2x)^b(2-2x)^a \leq 2$ then assuming $a \leq b$ (due symmetry of inequality) and, denoting $t := 1-2x, h := b-a$, we can rewrite original problem in the following,more convenient for further, equivalent form:

Let a and h be real numbers such that $a > 0, h \geq 0$ and $a \geq \dfrac{h(h-1)}{2}$. Prove that

(**1**) $\quad (1-t^2)^a((1+t)^h + (1-t)^h) \leq 2$ for $|t| \leq 1$, with equality if and only if $x = 0$.

Note that we can assume that $h > 1$, because otherwise, if $h \leq 1$ then by PM-AM inequality

$\left(\dfrac{(1+t)^h + (1-t)^h}{2}\right)^{1/h} \leq \dfrac{(1+t)+(1-t)}{2} = 1 \Leftrightarrow (1+t)^h + (1-t)^h \leq 2$ and, since $(1-t^2)^a \leq 1$, we obtain $(1-t^2)^a((1+t)^h + (1-t)^h) \leq 2$. So, let now $h > 1$

Since left hand side of inequality (**1**) is even function of t and equal to zero if $t = 1$ we can for further assume that $t \in [0,1)$ and rewrite inequality as

(**2**) $(1+t)^h + (1-t)^h \leq 2(1-t^2)^{-a}$, $a > 0$ and $h > 1$.

Using binomial series for $(1-t^2)^{-a}, (1+t)^h, (1-t)^h$ we obtain

$(1-t^2)^{-a} = 1 + \sum\limits_{n=1}^{\infty} a_n t^{2n}$ and $(1+t)^h + (1-t)^h = 2\left(1 + \sum\limits_{n=1}^{\infty} b_n t^{2n}\right)$, where

$a_n := (-1)^n \binom{-a}{n} = \dfrac{a(a+1)\ldots(a+n-1)}{n!}$ and $b_n := \binom{h}{2n} = \prod\limits_{k=1}^{n} \dfrac{(h-2k+2)(h-2k+1)}{(2k-1)2k}$,

Thus, inequality (**2**) becomes $\sum\limits_{n=1}^{\infty} a_n t^{2n} \leq \sum\limits_{n=1}^{\infty} b_n t^{2n}$ and remains to prove that $a_n \leq b_n$ for any $n \in \mathbb{N}$.

Since $a > 0$ then $a_n > 0$ for any $n \in \mathbb{N}$. But behavior of the sign b_n is more complicated.

If $h \in (2m-1, 2m)$ for some natural $m \geq 2$ then $b_n > 0$ for any $n \in \mathbb{N}$ because $(h-2k+2)(h-2k+1) > 0$ for any $k \in \mathbb{N}$;

If $h \in [2m, 2m+1]$ then $b_n > 0$ for any $n \leq m$ and $b_n \leq 0$ for any $n > m$ because $(h-2m)(h-2m-1) \leq 0$ and $(h-2k+2)(h-2k+1) > 0$ for any $k > m$. In that case suffice to prove that $a_n \geq b_n$ for any $n \leq m$.

Let $I(h) := \mathbb{N}$ if $h \in (2m-1, 2m)$ and $I(h) = \{1, 2, .., m\}$ if $h \in [2m, 2m+1]$

We will prove that $a_n \geq b_n$ for any $n \in I(h)$ using Math Induction with base $a_1 \geq b_1 \Leftrightarrow a \geq \dfrac{h(h-1)}{2}$.

Step of Math Induction:

For any $n \in I(h)$ and $n \geq 2$ let $a_{n-1} \geq b_{n-1}$. We will prove that

$$\frac{a_n}{a_{n-1}} \geq \frac{b_n}{b_{n-1}} \Leftrightarrow \frac{(-1)^n\binom{-a}{n}}{(-1)^{n-1}\binom{-a}{n-1}} \geq \frac{\binom{h}{2n}}{\binom{h}{2n-2}} \Leftrightarrow$$

$$\frac{a+n-1}{n} \geq \frac{(h-2n+2)(h-2n+1)}{2n(2n-1)} \Leftrightarrow a+n-1 \geq \frac{(h-2n+2)(h-2n+1)}{2(2n-1)}.$$

Since $a \geq \frac{h(h-1)}{2}$ we have $a+n-1-\frac{(h-2n+2)(h-2n+1)}{2(2n-1)} \geq$

$$\frac{h(h-1)}{2}+n-1-\frac{(h-2n+2)(h-2n+1)}{2(2n-1)} = \frac{h(h+1)(n-1)}{2n-1} > 0.$$

Hence, $a_n = a_{n-1} \cdot \frac{a_n}{a_{n-1}} \geq b_{n-1} \cdot \frac{b_n}{b_{n-1}} = b_n$.